Strong IP Formulations Need Large Coefficients

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Example I – The Good

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$$X = \left\{ x \in \{0,1\}^n \ : \ \sum_{i=1}^{n} 2^{n-i} x_i \leq 2^n - 3 \right\}$$

Pro:

- small IP formulation

Con:

- badly scaled coefficients
  $\rightsquigarrow$ numerical instabilities

Question: Is there a better IP formulation?

- tightest IP formulation consists of facet inequalities of conv(X)
- complete linear description [Laurent, Sassano 1992]

$$\left\{ x \in [0,1]^n \ : \ \sum_{i=1}^{n-1} x_i \leq n - 2 \right\}$$

- facet description is numerically stable

Example II – The Bad

$$X = \left\{ x \in \{0,1\}^{2n} \ : \ \sum_{i=1}^{n} 2^{n-i}(x_i + x_{n+i}) \leq 2^n - 1 \right\}$$

Question: Can we use facet description of conv(X)?

- consists of $\Theta(3^n)$ inequalities [Kaibel, Loos 2011]
- separable in linear time [Loos 2010]
- contains badly scaled inequalities, e.g.,

$$x_n + x_{2n} + \sum_{i=1}^{n-1} 2^{n-1-i}(x_i + x_{n+i}) \leq 2^{n-1}$$

- facet description is numerically instable

Questions

Every $X \subseteq \{0,1\}^n$ admits IP formulation with $\{0,\pm 1\}$-inequalities, e.g., via infeasibility cuts

$$\sum_{i=1}^{n} \Big((1-\bar{x}_i)\, x_i + \bar{x}_i\, (1-x_i) \Big) \geq 1 \qquad \forall \bar{x} \in \{0,1\}^n \setminus X,$$

but

1. Are such IP formulations strong?
2. If not, what is the minimum size of coefficients in strong IP formulations?

Outline

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Measure of Size of Coefficients

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Inequality $a^\top x \leq \beta$ is numerically more stable the smaller the ratio

$$\rho(a) \coloneqq \max\left\{\frac{|a_i|}{|a_j|} \;:\; a_j \neq 0,\; i \neq j\right\}.$$

Definition

The ρ-value of an IP formulation $Ax \leq b$ is

$$\max\{\rho(a) \;:\; a \text{ is row of } A\}.$$

$\rightsquigarrow$ the smaller the ρ-value the higher the numerical stability

Strong IP Formulations

Question: How to measure strength of IP formulation $Ax \leq b$?

- IP formulation of $X \subseteq \{0,1\}^n$ has to cut off points $x \in \mathbb{Z}^n \setminus X$
- strongest formulation uses facets of $\operatorname{conv}(X)$ cutting off all points in $\mathbb{R}^n \setminus \operatorname{conv}(X)$

Idea: Refine IP formulations by not only cutting of infeasible binary points but also points in a refinement of the integer lattice.

Definition

Let $\lambda \in \mathbb{Z}_{>0}$ and $X \subseteq \{0,1\}^n$. Then $Ax \leq b$ is called $\frac{1}{\lambda}$-relaxation of X if for every $\bar{x} \in \frac{1}{\lambda}\mathbb{Z}^n$ we have $A\bar{x} \leq b$ iff $\bar{x} \in \operatorname{conv}(X)$.

Measuring Strength of IP Formulations

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Measure of Strength

If $Ax \leq b$ is a $\frac{1}{\lambda}$-relaxation of X, then it is the stronger the larger λ.

In particular, $\frac{1}{\lambda}$-relaxations might have small coefficients, while facets of conv(X) have large coefficients.

But: How to estimate the size of coefficients in $\frac{1}{\lambda}$-relaxations?

Observation

$$
\begin{aligned}
& Ax \leq b,\ x \in [0,1]^n \text{ is } \tfrac{1}{\lambda}\text{-relaxation of } X \\
\Leftrightarrow\quad & Ax \leq \lambda\, b,\ x \in [0,\lambda]^n \text{ is IP formulation of } (\lambda \operatorname{conv}(X)) \cap \mathbb{Z}^n.
\end{aligned}
$$

$\rightsquigarrow$ Find bounds on coefficients in general IP formulations.

Lower Bound on Coefficients

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Theorem [H. 2018]

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional integral polytope. Let

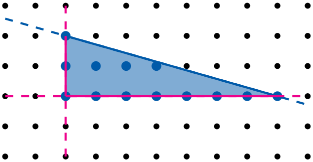
- $\bar{x} \in \mathbb{Z}^n \setminus P$ be not cut off by any valid box constraint.
- $\bar{A}x \leq \bar{b}$ consist of all facet inequalities for P cutting off $\bar{x}$.
- $\bar{s} \coloneqq \bar{A}\bar{x} - \bar{b}$.

If some technical assumptions hold, every valid inequality $c^\top x \leq \delta$ cutting off $\bar{x}$ fulfills

$$\frac{|c_i|}{|c_j|} \geq \frac{\min_k\{|\bar{A}_{ki}| - \bar{s}_k\}}{\max_k\{|\bar{A}_{kj}| + \bar{s}_k\}}.$$

Proof Idea

Consider $P = \text{conv}\{x \in \mathbb{R}^2 \ : \ 2\,x_1 + 7\,x_2 \leq 14, x_1 \geq 0, x_2 \geq 0\}$.



Question: Is there an IP formulation with smaller ρ-value?

Idea:

- select $\bar{x} \in \mathbb{Z}^2 \setminus P$
- derive bounds on coefficients in any valid inequality cutting of $\bar{x}$

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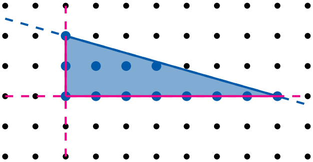

Question: Is there an IP formulation with smaller ρ-value?

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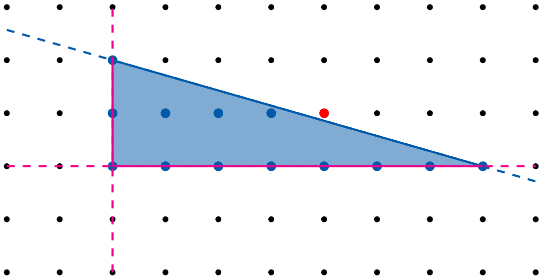
- derive bounds on coefficients in any valid inequality cutting of $\bar{x}$

Lemma

If P is a full-dimensional polytope, every valid inequality for P is a conic combination of facet defining inequalities.

Example

Consider $P = \text{conv}\{x \in \mathbb{R}^2 \ : \ 2\,x_1 + 7\,x_2 \leq 14, x_1 \geq 0, x_2 \geq 0\}$ and select $\bar{x} = \binom{4}{1}$.



Example

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Consider $P = \operatorname{conv}\{x \in \mathbb{R}^2 \ : \ 2\,x_1 + 7\,x_2 \leq 14, x_1 \geq 0, x_2 \geq 0\}$ and select $\bar{x} = \binom{4}{1}$.

- up to scaling, every valid inequality has form

$$c^\top x \leq \delta \qquad :\Leftrightarrow \qquad (2-\alpha)x_1 + (7-\beta)x_2 \leq 14$$

- inserting $\bar{x}$ yields

$$(2-\alpha)\bar{x}_1 + (7-\beta)\bar{x}_2 > 14 \qquad \Leftrightarrow \qquad 4\alpha + \beta < 1$$

- together with $\alpha \geq 0$ and $\beta \geq 0$

$$\rho(c) = \frac{7-\beta}{2-\alpha} > \frac{6+4\alpha}{2-\alpha} \geq \frac{6}{2} = 3$$

Generalization to General Integer Polytopes

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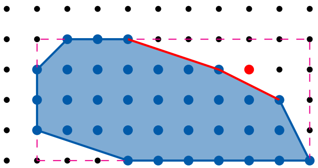

Notation:

- B smallest box containing P with lower bounds ℓ and upper bounds u
- $\bar{x} \in (B \setminus P) \cap \mathbb{Z}^n$
- $\bar{A}x \leq \bar{b}$ facets violated by $\bar{x}$
- $Ax \leq b$ remaining facets

Idea: To find lower bounds on the ρ-value, use the previous lemma to find upper and lower bounds on coefficients in inequalities cutting of given $\bar{x}$.

Problem: Bounds depend on conic multipliers.
$\rightsquigarrow$ Introduce further conditions to get rid of multipliers.

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Condition I to Bound $\frac{|c_i|}{|c_j|}$

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Sign Restrictions:

- For all rows k, k' of $\bar{A}$, we have

$$\operatorname{sgn}(\bar{A}_{kt}) = \operatorname{sgn}(\bar{A}_{k't}) \neq 0 \qquad \forall t \in \{i, j\}.$$

- For every row k of A not corresponding to a box constraint, we have

$$\operatorname{sgn}(A_{kt}) \in \{0, \operatorname{sgn}(\bar{A}_{1t})\} \qquad \forall t \in \{i, j\}.$$

$\bar{x} \in [\ell, u] \setminus P$ $\quad$ $\bar{A}x \leq \bar{b}$ facets violated by $\bar{x}$ $\quad$ $Ax \leq b$ remaining facets $\quad$ $\bar{s} := \bar{A}\bar{x} - \bar{b}$

Condition II to Bound $\frac{|c_i|}{|c_j|}$

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Box Restrictions:

- $\bar{x}_i > \ell_i$.
- $\bar{x}_j < u_j$.
- For every inequality $a^\top x \leq \beta$ in $Ax \leq b$ not corresponding to a box constraint, we have

$$\sum_{t=1}^{n} a_t \bar{x}_t + \operatorname{sgn}(\bar{A}_{1j})\, e_j \leq \beta.$$

$\bar{x} \in [\ell, u] \setminus P$ $\quad$ $\bar{A}x \leq \bar{b}$ facets violated by $\bar{x}$ $\quad$ $Ax \leq b$ remaining facets $\quad$ $\bar{s} \coloneqq \bar{A}\bar{x} - \bar{b}$

Discrete Optimization

Condition III to Bound $\frac{|c_i|}{|c_j|}$

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Excess Restrictions:

- $|\bar{A}_{ki}| \geq \bar{s}_k$ for every row k of $\bar{A}$.
- $\bar{x}_j - \ell_j \geq \max_k\{\bar{s}_k\}$.

$\bar{x} \in [\ell, u] \setminus P$ $\quad$ $\bar{A}x \leq \bar{b}$ facets violated by $\bar{x}$ $\quad$ $Ax \leq b$ remaining facets $\quad$ $\bar{s} := \bar{A}\bar{x} - \bar{b}$

Lower Bound on Coefficients

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Theorem [H. 2018]

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional integral polytope. Let

- $\bar{x} \in \mathbb{Z}^n \setminus P$ be not cut off by any valid box constraint.
- $\bar{A}x \leq \bar{b}$ consist of all facet inequalities for P cutting off $\bar{x}$.
- $\bar{s} \coloneqq \bar{A}\bar{x} - \bar{b}$.

If Conditions I–III hold, every valid inequality $c^\top x \leq \delta$ cutting off $\bar{x}$ fulfills

$$\frac{|c_i|}{|c_j|} \geq \frac{\min_k\{|\bar{A}_{ki}| - \bar{s}_k\}}{\max_k\{|\bar{A}_{kj}| + \bar{s}_k\}}.$$

Applications: IP formulations

Using this theorem, one can show:

- Every IP formulation of

$$\left\{ x \in \mathbb{Z}_+^{n+1} \ : \ \sum_{i=0}^{n} 2^i x_i \leq 2^n \right\}$$

  has ρ-value at least $\frac{2^n - 1}{2}$.
- Consequently, there exist $X \subseteq \mathbb{Z}^n$ that need exponentially large coefficients in any IP formulation.

Applications: $\frac{1}{\lambda}$-relaxation

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Recall

$Ax \leq b$ is $\frac{1}{\lambda}$-relaxation of X $\Leftrightarrow$ $Ax \leq \lambda b$ is IP formulation of $(\lambda \operatorname{conv}(X)) \cap \mathbb{Z}^n$.

- The theorem implies every $\frac{1}{3}$-relaxation of

$$\Big\{x \in \{0,1\}^{3n+2} : x_{3n+1} + x_{3n+2} + \sum_{i=1}^{n} 2^{3(n-i)+4} x_{3(i-1)+1} + \sum_{i=1}^{n} 2^{3(n-i)+3)} x_{3(i-1)+2} + \sum_{i=1}^{n} (2^{3(n-i)+4} - 2^{3(n-i)+2}) x_{3i} \leq 1 + \sum_{i=1}^{n} (2^{3(n-i)+4} - 2^{3(n-i)+2}) \Big\}$$

  has ρ-value at least $\frac{3^{n/3-2}-1}{2}$.
- result can be generalized to exponentially large class of knapsacks

Consequences

Corollary

There exist sets $X \subseteq \{0,1\}^n$ that need exponentially large coefficients in every $\frac{1}{\lambda}$-relaxation if $\lambda \geq 3$.

Remark: result can be generalized to case $\lambda \geq 2$

Consequences:

- every $X \subseteq \{0,1\}^n$ admits 1-relaxation with $\{0, \pm 1\}$-coefficients.
- $\frac{1}{2}$-relaxations may need large coefficients
- strong IP formulations need exponentially large coefficients in general

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Thank You For Your Attention